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Fixed point resolution in extended WZW-models

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Abstract

A formula is derived for the fixed point resolution matrices of simple current extended WZW-models and coset conformal field theories. Unlike the analogous matrices for unextended WZW-models, these matrices are in general not symmetric, and they may have field-dependent twists. They thus provide non-trivial realizations of the general conditions presented in earlier work with Fuchs and Schweigert.

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1. Introduction

Despite a large amount of work on conformal field theory during the last fifteen years, there are still no efficient procedures for computing many quantities of interest. Examples of such quantities are the spectrum, the (Virasoro or extended) characters, the modular transformation properties of these characters, the fusion rules, fusing and braiding matrices, and correlation functions on arbitrary Riemann surfaces. While the difficulties in computing the latter quantities are well-known, even the determination of the spectrum can involve unexpected difficulties that are often overlooked in the literature. Indeed, for a large class of rational CFT's that are very explicitly defined, namely the class of unitary coset conformal field theories, it is not even known *in general* how to compute the spectrum (although in certain simple cases no problems occur). Even determining the correct *number* of primaries in such theories – not to mention the conformal weights – is non-trivial in some classes of coset theories and extended WZW-models, and was only solved a few years ago in [1]. The determination of the characters – and hence the spectrum – is only fully known for the special case of diagonal coset theories [2].

The problems alluded to in the end of the previous paragraph all have a common origin, that of resolving fixed points in simple current extensions. This problem arises in many coset CFT's (as was first noted in [3] and [4]) due to the procedure known as "field identification" [5], which is formally analogous to a simple current extension of the chiral algebra [6]. In general – whether applied to the construction of extended chiral algebras or to field identification – simple currents may have fixed points which must be resolved in order to arrive at a correct description of the theory. This fixed point resolution procedure requires data in addition to the characters (or branching functions) and modular matrix S of the unextended theory. This information is then used to modify the coset branching functions to obtain the characters, and to modify the "naive" matrix S to obtain the correct one.

It is the latter aspect of fixed point resolution that interests us here. The additional information needed to resolve the fixed points are the "fixed point resolution matrices" S^J, which exist for any simple current J that has fixed points. In [1] these matrices were obtained for unextended WZW-models ("A-invariants"). They turned out to be simply related to ordinary modular transformation matrices S of an "orbit Lie algebra" [7] related to the WZW-model. In general the matrices S^J have been conjectured [1] to be related to the modular transformation matrices of holomorphic one-point blocks [8]. Strong evidence for this conjecture was provided in [9].

In this paper we will give a formula for the fixed point resolution matrices of simple current extended rational conformal field theories. These matrices are expressed in terms of the corresponding matrices of the unextended CFT. Although *a priori* nothing restricts us to WZW-models, this is the only class for which the fixed point resolution matrices are already known, and hence the obvious starting point. The main application of our result is therefore to simple current extended WZW-models (also known as D-invariants) and field identification in coset theories. This then generalizes the results of

[1] to all coset theories, except a small class described in [10], which has "exceptional" field identifications. Since most rational CFT's can be described as coset CFT's, this is a step in the direction of generalizing the results of [1] to all rational CFT's, and allows us to examine whether, and how, the general case differs from the special case of WZW-models.

In [1] a set of conditions for fixed point resolution matrices in generic rational CFT's was presented. The fixed point resolution matrices of unextended WZW-models were found to satisfy additional constraints, due to the fact that they are modular transformations of an orbit Lie algebra. We find that in extended WZW-models most of these additional constraints do not hold, and only the necessary conditions are satisfied. In particular the fixed point resolution matrices do not appear to be related to any kind of orbit Lie algebra. This concept apparently does not generalize to conformal field theory.

The fixed point resolution matrices S^J are needed to compute the modular transformation matrix S of simple current extended CFT's. Using the Verlinde formula one may then compute the fusion rules. The results presented here allow the computation of S if a simple current extended CFT is extended once more by another set of simple currents. The result of this procedure is an enlargement of the original chiral algebra in two steps. Obviously, the final result can also be obtained by making the entire extension in just one step, and in that case the formula of [1] already gives the matrix S of the twice-extended theory in terms of the fixed point resolution matrices of the unextended theory. Hence the fixed point resolution matrices of the once-extended theory are not strictly needed for the purpose of computing S (although they are in some cases convenient for practical reasons). However, it has recently become clear that the fixed point resolution matrices appear also in other contexts [11], namely in the description of boundary conditions for open strings built as "descendants" of certain conformal field theories [12]. For these applications we need to know these matrices explicitly.

Since the general formula of [1] covers the first extension, the second extension as well as the full extension, it is clear that the information we are looking for can be derived from the results of that paper. Indeed, this fact played an important rôle in obtaining the general formula and the conditions governing S^J, and some cases of successive extensions are discussed in [1]. It turns out, however, that there are several subtleties in the general case. After treating these carefully, we finally arrive at a formula for S^J. The fixed point resolution matrices S^J may be viewed as a generalization of the modular transformation matrix $S \equiv S^1$. It is then not surprising that the formula we obtain for S^J is quite similar to the one obtained for S in [1]. The main difference consists of some additional phase factors that vanish in the special case of S^1, but are essential for consistency of S^J.

This paper is organized as follows. In the next section we summarize the results of [1], with a minor improvement in the formulation of the conditions for S^J. The conditions involve two new quantities, the "simple current twist" $F(a, K, J)$ and the "conjugation twist" $G(a, K, J)$, where a is a primary field fixed by the simple currents K and J. These quantities are equal to each other for unextended WZW-models (and

independent of a). We give an argument for the validity of this equality in all CFT's with real twists (which so far are the only ones known). The details are given in Appendix A. On the other hand, a-independence turns out *not* to be a general feature, as is shown by means of an example in section 5. In section 3 we give the precise formulation of the problem we want to solve, and we also discuss the choice of orbit representatives needed in the derivation. The existence proof for our choice of representatives is presented in Appendix B. The derivation of the main result is presented in section 4. It involves two main ingredients: the proper choice of representatives, and the splitting of discrete group characters. The mathematical background for this splitting is provided in Appendix C. Both of these ingredients lead to crucial phase factors in the main result, formula 4.4. In section 5 we investigate some of the properties of the matrices S^J.

2. Fixed point resolution

In this section we summarize the results of [1]. The reader is assumed to be familiar with simple currents and fixed points (see [13] for a review).

Consider a rational CFT C with a set of mutually local, integer spin simple currents forming a subgroup $\mathcal{H}$ of the full center $\mathcal{G}$. We extend the chiral algebra by the currents in $\mathcal{H}$, and we denote the new CFT as $\mathrm{C}^{\mathcal{H}}$. The fields in the new theory correspond to $\mathcal{H}$-orbits of fields* of C that are local with respect to $\mathcal{H}$, but in general this is not a one-to-one correspondence. For each field a of C we define the stabilizer $\mathcal{S}_a$ as the set of currents J in $\mathcal{H}$ that fixes a: $Ja = a$. Furthermore we define a certain subgroup of $\mathcal{S}_a$, the untwisted stabilizer $\mathcal{U}_a$. The details of this definition follow below. Then the fields of $\mathrm{C}^{\mathcal{H}}$ are labelled by a pair (a, i), where a is a representative of an $\mathcal{H}$-local orbit, and i is a character label of $\mathcal{U}_a$. The modular transformation matrix of $\mathrm{C}^{\mathcal{H}}$ is given by [1]

$$S_{(a,i)(b,j)} = \frac{|\mathcal{H}|}{\sqrt{|\mathcal{S}_a||\mathcal{U}_a||\mathcal{S}_b||\mathcal{U}_b|}} \sum_J \Psi_i^J S_{ab}^J (\Psi_j^J)^* \ , \tag{2.1}$$

where the sum is over the intersection of the two untwisted stabilizers. The characters should carry an extra label a or b to indicate to which untwisted stabilizer they belong, but this label is always already implied by the fixed point resolution labels, and will therefore be suppressed as much as possible.

The matrices S_{ab}^J appearing in this formula must satisfy the following properties

- {1} $\quad S_{ab}^J = 0$ if $Ja \neq a$ or $Jb \neq b$
- {2} $\quad S^J (S^J)^\dagger = 1$
- {3} $\quad (S^J T^J)^3 = (S^J)^2$
- {4} $\quad S_{Ka,b}^J = F(a, K, J) e^{2\pi i Q_K(b)} S_{ab}^J$

$\star$ The term "field" is used throughout this paper as an abbreviation for "primary field"

- $\{4a\}$ $F(a,K,J_1)F(a,K,J_2) = F(a,K,J_1J_2)$
- $\{5\}$ $(S^J)^2 = \eta^J C^J$
- $\{5a\}$ $\eta^J_{ab} = \eta^J_a \delta_{ab}$
- $\{5b\}$ $\eta^J_a \eta^K_a = G(a,K,J)\eta^{JK}_a$, $G(a,K,J) = 1$ if $F(a,K,J) = 1$
- $\{5c\}$ $\eta^J C^J = C^J (\eta^J)^*$
- $\{6\}$ $S^J_{ab} = S^{J^{-1}}_{ba}$

Here properties $\{2\} - \{6\}$ are defined on the subspace of fields where S^J is non-zero, and T^J, C^J are the restrictions of T and C to that subspace; as usual T is the generator of the $\tau \to \tau + 1$ transformation of the modular group, S is the generator of $\tau \to -\frac{1}{\tau}$, and C is the charge conjugation matrix. These three matrices satisfy $(ST)^3 = S^2 = C$. The quantity $Q_J(a)$ is the monodromy charge of the field a w.r.t. the current J. The untwisted stabilizer $\mathcal{U}_a$ is defined as

$$\mathcal{U}_a := \{J \in \mathcal{S}_a \mid F(a,K,J) = 1 \text{ for all } K \in \mathcal{S}_a\}$$

.

In comparison to [1] the condition $\{5b\}$ has been formulated differently. In [1] it was given as $\eta^J_a \eta^K_a = \eta^{JK}_a$ if $J, K \in \mathcal{U}_a$. It is easy to see that both forms of the condition are equivalent, but the form chosen here has the advantage that no explicit mention is made of (untwisted) stabilizers in the conditions for S^J. Indeed, the matrices S^J are a property of the CFT C and do not depend on the extension of the chiral algebra that one considers, and hence in particular not on $\mathcal{S}_a$ and $\mathcal{U}_a$.

One can derive from these conditions that the simple current twists $F(a,K,J)$ are phases satisfying

$$F(a,K_1,J)F(a,K_2,J) = F(a,K_1K_2,J) \tag{2.2}$$

and

$$F(a,K,J) = F(a,J,K)^* \; , \tag{2.3}$$

provided that $Ka = a$ (note that the definition $\{4\}$ of F requires J, but not K, to fix a).† Furthermore one can show that $F(a,J,J) = 1$ if J has integer spin, and $F(a,J,J) = -1$ if J has half-integer spin.

In the special case of unextended WZW-models the matrices S^J were obtained in [1]. They have some additional properties not required for fixed point resolution, and in particular they are themselves modular transformation matrices of a set of characters of (in some cases twisted) affine Lie algebras. Therefore they are symmetric, and $S^J = S^{J'}$

† If $Ka \neq a$ the twists depend on a fixed point labelling convention: one may label the resolved fixed points of a differently than those of Ka.

if J and J' generate the same cyclic subgroup (hence $S^J = S^{J^{-1}}$). Furthermore η_a^J and $F(a,K,J)$ are independent of a. For unextended WZW-models the twists satisfy the empirical relation

$$G(a,K,J) = F(a,K,J)\ . \tag{2.4}$$

which implies $\{5b\}$, but is stronger.

Note that (2.4) implies that F is real, *i.e.* $F = \pm 1$, since G is symmetric in J and K, and F is symmetric only if it is real. Under a very mild additional assumption the converse is also true, *i.e.* reality of F implies (2.4). The additional assumption is

— The matrices S^J of tensor product CFT's are the tensor products of the matrices S^J of the factors.

The proof of the converse statement (*i.e.* reality of F implies $F = G$) is given in appendix A. In all known cases F is real, and (2.4) holds. It would be interesting to find a fundamental reason why either property should hold in general, but at least this argument shows that they are related.

3. Formulation of the problem

3.1. Definitions

Consider a CFT C with center $\mathcal{G}$. The fields of the CFT will be denoted by a. For each field a there is a subgroup of $\mathcal{G}$ that fixes a. This group will be called the *full stabilizer* of a, and will be denoted as $\mathcal{T}_a$. It depends in fact only on the $\mathcal{G}$-orbit of a, not on orbit members individually. It consists of currents of integer or half-integer spin.

For each current J in $\mathcal{G}$ we assume the existence of a matrix S^J_{ab} satisfying the conditions of [1], stated in the previous section.

We will consider here new conformal field theories obtained by extending the chiral algebra of C by means of a subgroup $\mathcal{H}$ of $\mathcal{G}$ consisting of mutually local integer spin currents. The new theory will be called $\mathrm{C}^{\mathcal{H}}$. In general, all quantities in the extended theory will be denoted by superscripts $\mathcal{H}$. For example, $a^{\mathcal{H}}$ denotes an $\mathcal{H}$ primary. The center of the new theory will be denoted as $\mathcal{G}^{\mathcal{H}}$. It contains the subgroup of $\mathcal{G}$ of currents local w.r.t. $\mathcal{H}$, organized into $\mathcal{H}$-orbits.‡ To determine the spectrum and the modular transformation matrix of $\mathrm{C}^{\mathcal{H}}$ one needs the stabilizer of each field. This is the set of currents in $\mathcal{H}$ that fixes a. Hence $\mathcal{S}_a = \mathcal{T}_a \cap \mathcal{H}$. This stabilizer should not be confused with the full stabilizer $\mathcal{T}^{\mathcal{H}}_{a^{\mathcal{H}}}$ of the field $a^{\mathcal{H}}$ of $\mathrm{C}^{\mathcal{H}}$, consisting of the currents in $\mathcal{G}^{\mathcal{H}}$ that fix $a^{\mathcal{H}}$. Another important group is the untwisted stabilizer, whose characters labels the resolved fixed points. More precisely, the fields $a^{\mathcal{H}}$ are labelled by a set (a,i), where i labels a representation of $\mathcal{U}_a$ and a is a representative of a $\mathcal{H}$-local $\mathcal{H}$-orbit.

‡ In rare cases $\mathcal{G}^{\mathcal{H}}$ may be larger due to extra currents originating from resolved fixed points, but we will ignore such currents here.

Our goal is the computation of the matrices $S^{J^{\mathcal{H}}}_{a^{\mathcal{H}}b^{\mathcal{H}}}$ and the determination of the twists $F^{\mathcal{H}}(a^{\mathcal{H}}, K^{\mathcal{H}}, J^{\mathcal{H}})$.

3.2. Choice of representatives

Consider the action of the simple current $J^{\mathcal{H}}$ on a resolved fixed point field $a^{\mathcal{H}} = (a, i)$. The elements of the corresponding coset are $\{J, Jh_1, \dots Jh_n\}$, where J is some $\mathcal{H}$-local element of $\mathcal{G}$ and h_i denote the elements of $\mathcal{H}$. The current J is the representative chosen to identify the orbit. Suppose that in the $\mathcal{H}$-*extended* theory the current $J^{\mathcal{H}}$ fixes $a^{\mathcal{H}}$. Then J must map a to another member of the $\mathcal{H}$ orbit of a, *i.e.* $Ja = h_i a$ for some $h_i \in \mathcal{H}$. Then Jh_i^{-1} fixes a. Hence we can always find a representative that fixes a, and we will denote this representative as $X_a(J)$, $X_a(J) \in J\mathcal{H}$. In general X_a depends on a, and there is no universal choice of representatives that avoids such a dependence for all currents. The complete set of C-currents in the coset element $J^{\mathcal{H}}$ that fixes a is of the form

$$X_a(J)\mathcal{S}_a \ .$$

The existence of such a representative X_a is necessary, but not sufficient to conclude that $J^{\mathcal{H}}$ fixes $a^{\mathcal{H}}$. It only ensures that $J^{\mathcal{H}}$ fixes the first of the two labels (a, i). The action of $J^{\mathcal{H}}$ on the second label is trivial if and only if

$$F(a, X_a(J), K) = 1 \text{ for all } K \in \mathcal{U}_a \tag{3.1}$$

This condition is the same for all choices $X_a(J)\mathcal{S}_a$, because (using (2.2) and the definition of the untwisted stabilizer)

$$F(a, X_a h, K) = F(a, X_a, K)F(a, h, K) = F(a, X_a, K) \ ,$$

for all $h \in \mathcal{S}_a$.

If (3.1) holds it is always possible to make a choice $R_a(J)$ out of the set of representatives $X^a(J)\mathcal{S}_a$ such that

$$F(a, R_a(J), K) = 1 \text{ for all } K \in \mathcal{S}_a \tag{3.2}$$

This choice R_a is unique up to multiplication by elements of $\mathcal{U}_a$ and is essential in the derivation in the next section. The proof that such a choice is always possible is given in appendix B.

4. Derivation of the main result

The FCFT-matrix for a current $J^{\mathcal{H}}$ can in principle be computed from the general formula. Instead of extending $\mathrm{C}^{\mathcal{H}}$ we may extend C by a set of simple currents $\mathcal{M} \subset \mathcal{G}$ in such a way that $\mathcal{M}$ contains all $\mathcal{G}$-elements of the form $J^n\mathcal{H}$ (where J is a representative of the coset corresponding to $J^{\mathcal{H}}$). Eqn. (2.1) is in this case

$$S_{(a,\alpha)(b,\beta)} = \frac{|\mathcal{M}|}{\sqrt{|\hat{\mathcal{S}}_a||\hat{\mathcal{U}}_a||\hat{\mathcal{S}}_b||\hat{\mathcal{U}}_b|}} \sum_L \Xi^L_\alpha S^L_{ab} (\Xi^L_\beta)^* \ , \tag{4.1}$$

where the sum is over the intersection of the two untwisted stabilizers. The hats on $\hat{\mathcal{S}}$ and $\hat{\mathcal{U}}$ distinguish these groups from the (untwisted) stabilizers in the $\mathcal{H}$ extension, and we will use the symbol Ξ to denote $\hat{\mathcal{U}}$ characters. The $\mathcal{U}$ characters will be denoted by Ψ. The labels α and β distinguish the resolved fixed points. For the resolved fixed points in the $\mathcal{H}$-extended theory we will use labels i and j.

Let N be the smallest integer larger than zero so that $J^N \in \mathcal{H}$. Then $|\mathcal{M}| = N|\mathcal{H}|$. As was shown in the previous section, the field labelled $a^{\mathcal{H}}$ is a fixed point if and only if there exists a current $h \in \mathcal{H}$ so that $Jha = a$, and $F(a, Jh, K) = 1$ for all $K \in \mathcal{U}_a$. If such an element h exists we may in fact choose a class representative $R_a(J)$ so that $R_a(J)a = a$ and $F(a, R_a(J), K) = 1$ for all $K \in \mathcal{S}_a$. Clearly $\hat{\mathcal{S}}_a$ consists of N classes $(R_a(J))^n\mathcal{S}_a$, $n = 0, \ldots, N-1$, so that $|\hat{\mathcal{S}}_a| = N|\mathcal{S}_a|$.

Also the untwisted stabilizer $\mathcal{U}$ gets extended by $R_a(J)$ and all its powers, because (omitting the arguments a and J for simplicity)

$$F(R^nK, R^mK') = F(R^n, R^m)F(K, R^m)F(R^n, K')F(K, K') = F(K, K') \ ,$$

for all K and $K' \in \mathcal{S}_a$. Here we used the group property in both arguments. The result is equal to 1 if (and only if) in the second argument K' is restricted to $\mathcal{U}$. This shows that R^mK' is untwisted with respect to $\hat{\mathcal{S}}_a$ and hence is in $\hat{\mathcal{U}}_a$.

Hence in any case of interest to us $|\hat{\mathcal{U}}_a| = N|\mathcal{U}_a|$.

It should be noted that there are several other situations possible that are not of immediate interest. For example, on some fields the untwisted stabilizer may decrease rather than increase in size. Then the current $J^{\mathcal{H}}$ acts non-trivially on the fixed point resolution labels and re-combines several resolved fixed points into a single field. The stabilizer $\mathcal{S}_a$ on the other hand can never get smaller, but it can be enlarged by non-trivial divisors of N. This happens if $J^{\mathcal{H}}$ does not fix $a^{\mathcal{H}}$, but some power (smaller than N) does fix $a^{\mathcal{H}}$. However, in neither of these cases $a^{\mathcal{H}}$ is fixed by $J^{\mathcal{H}}$, and hence we do not have to consider these possibilities.

Given (4.1) we can obtain the fixed point resolution matrix $J^{\mathcal{H}}$ by "pulling out" the $\mathbf{Z}_N$ characters corresponding to the extension by $J^{\mathcal{H}}$. In general characters of a discrete group do not simply "factorize" in terms of characters of a subgroup. To do this correctly we need the formalism developed in Appendix C to write the characters of $\hat{\mathcal{U}}_a$ as a product of those of $\mathcal{U}_a$, those of $\mathbf{Z}_N$ and an additional phase factor.

4.1. Determination of the fixed point resolution matrix

Consider a $C^{\mathcal{H}}$-field $a^{\mathcal{H}}$. Out of the coset belonging to $a^{\mathcal{H}}$ we choose one representative a, which is a field in C. This choice is arbitrary. Now fix the representatives for all $C^{\mathcal{H}}$-currents that fix $a^{\mathcal{H}}$. This set of currents forms the full stabilizer $\mathcal{T}^{\mathcal{H}}_{a^{\mathcal{H}}}$. The representatives can be chosen by decomposing $\mathcal{T}^{\mathcal{H}}_{a^{\mathcal{H}}}$ into cyclic factors, and choosing representatives for the generator of each cyclic factor. We can choose these basis representatives $R_a(\text{basis})$ so that they fix the field a (rather than fixing it up to an element of $\mathcal{H}$). As was shown in Appendix B, we can also choose them so that the twist $F(a, R_a(\text{basis}), \mathcal{S}_a) = 1$. Both properties are preserved under multiplication of representatives. It follows then that the representatives $R_a(K)$ fix a for all $K^{\mathcal{H}} \in \mathcal{T}^{\mathcal{H}}_{a^{\mathcal{H}}}$, and that they are untwisted w.r.t. the stabilizer $\mathcal{S}_a$.

Having fixed the representatives we compute the phases ϕ that appear if we decompose $\mathcal{T}_a$ characters into $\mathcal{U}_a$ characters. Since $\mathcal{T}_a \supset \hat{\mathcal{U}}_a \supset \mathcal{U}_a$ this fixes the decomposition of all possible $\hat{\mathcal{U}}_a$ groups that may occur in various extensions of $C^{\mathcal{H}}$. When we extend by $J^{\mathcal{H}}$ we now have a canonical representative $R_a(J)$ available for each field $a^{\mathcal{H}}$ (and representative a). The phases ϕ depend on the choice of representatives R_a.

Using the results of appendix C we can write the characters of $\hat{\mathcal{U}}_a$ as follows

$$\Xi^{R_a(J^n)h}_{m,i} = \psi^n_m(\mathbf{Z}_N)\phi_i(n,a)\Psi^h_i(\mathcal{U}_a)$$

where $h \in \mathcal{U}_a$ and i labels the distinct $\mathcal{U}_a$ characters. To simplify the notation we write n instead of J^n in the upper index of ψ and the first argument of ϕ. In the following we simplify the notation further by dropping the arguments $\mathbf{Z}_N$ and $\mathcal{U}_a$, which are self-understood.

The formula for S in the $\mathcal{M}$-extended theory is (4.1), where α should be interpreted as (m, i). We can split the sum into a sum over J-cosets and a sum over $\mathcal{H}$:

$$S_{(a,\alpha)(b,\beta)} = \frac{|\mathcal{M}|}{\sqrt{|\hat{\mathcal{S}}_a||\hat{\mathcal{U}}_a||\hat{\mathcal{S}}_b||\hat{\mathcal{U}}_b|}} \sum_{n=0}^{N-1} \left[\sum_{h\in\mathcal{H}, J^n h \in \hat{\mathcal{U}}_a \cap \hat{\mathcal{U}}_b} \Xi^{J^n h}_{\alpha} S^{J^n h}_{ab} (\Xi^{J^n h}_{\beta})^* \right] .$$

Clearly there can only be a contribution if it is possible to choose $h \in \mathcal{H}$ in such a way that $J^n h$ is in the intersection of $\hat{\mathcal{U}}_a$ and $\hat{\mathcal{U}}_b$. We will denote this particular element $J^n h$ as $R_{ab}(J^n)$. The complete set of such elements is obtained by multiplying $R_{ab}(J^n)$ by any element of $\mathcal{U}_a \cap \mathcal{U}_b$. Hence we can now write the summation as

$$S_{(a,\alpha)(b,\beta)} = \frac{|\mathcal{M}|}{\sqrt{|\hat{\mathcal{S}}_a||\hat{\mathcal{U}}_a||\hat{\mathcal{S}}_b||\hat{\mathcal{U}}_b|}} \sum_{n=0}^{N-1} \left[\sum_{h\in\mathcal{U}_a \cap \mathcal{U}_b} \Xi^{R_{ab}(J^n)h}_{\alpha} S^{R_{ab}(J^n)h}_{ab} \right.$$
$$\left. (\Xi^{R_{ab}(J^n)h}_{\beta})^* \right] .$$

A legitimate choice for $R_{ab}(J^n)$ is $R_{ab}(J^n) = [R_{ab}(J)]^n$. However, the choice is irrelevant, since legitimate choices of $R_{ab}(J^n)$ differ by elements of $\mathcal{U}_a \cap \mathcal{U}_b$, over which we sum.

The representatives $R_{ab}(J^n)$ are valid choices for $\hat{\mathcal{U}}_a/\mathcal{U}_a$ as well as $\hat{\mathcal{U}}_b/\mathcal{U}_b$, but in general not equal to the choices $R_a(J^n)$ and $R_b(J^n)$ made *a priori*. We cannot allow the choice of representatives in $\hat{\mathcal{U}}_a/\mathcal{U}_a$ to depend on b. We must choose a fixed basis on each stabilizer, and refer everything to that basis. In other words, we must replace $R_{ab}(J^n)$ by either $R_a(J^n)$ or $R_b(J^n)$. To do this we use the group property

$$\Xi_\alpha^{R_{ab}(J^n)h} = \Xi_\alpha^{R_a(J^n)h}\Xi_\alpha^{(R_{ab}(J^n)/R_a(J^n))} \tag{4.2}$$

and the same for the index b. Now it should be noted that $R_{ab}(J^n)$ and $R_a(J^n)$ are both representatives of the same coset class, and hence their ratio is an element of $\mathcal{H}$, Furthermore, since both representatives fix a their ratio is in fact in $\mathcal{S}_a$. Finally, by construction both R_{ab} and R_a are untwisted with respect to $\mathcal{S}_a$, and therefore their ratio R_{ab}/R_a is an element of $\mathcal{U}_a$. Hence the second $\hat{\mathcal{U}}_a$ character Ξ_α in (4.2) reduces to a $\mathcal{U}_a$ character Ψ_i.

Now we wish to split off $\mathbf{Z}_N$ characters. Writing α as (m,i)

$$\Xi_\alpha^{R_a(J^n)h}\Psi_i^{(R_{ab}(J^n)/R_a(J^n))} = \psi_m^n\phi_i(n,a)\Psi_i^h\Psi_i^{(R_{ab}(J^n)/R_a(J^n))}$$

Substituting this into the general formula for S we find

$$S_{(a,m,i)(b,m',j)} = \frac{|\mathcal{M}|}{\sqrt{|\hat{\mathcal{S}}_a||\hat{\mathcal{U}}_a||\hat{\mathcal{S}}_b||\hat{\mathcal{U}}_b|}}\sum_{n=0}^{N-1}\Bigg[\sum_{h\in\mathcal{U}_a\cap\mathcal{U}_b} S_{ab}^{R_{ab}(J^n)h} \psi_m^n\phi_i(n,a)\Psi_i^{(R_{ab}(J^n)/R_a(J^n))h} [\psi_{m'}^n\phi_j(n,b)\Psi_j^{(R_{ab}(J^n)/R_b(J^n))h}]^*\Bigg]\,. \tag{4.3}$$

From this expression we read off the fixed point matrix for $J^n\mathcal{H}$, by removing the $\mathbf{Z}_N$ characters. The formula (4.1) for the $\mathbf{Z}_N$ extension of $\mathrm{C}^{\mathcal{H}}$, for matrix elements of untwisted fixed points of J (*i.e.* with $\mathcal{S}_a^{\mathcal{H}} = \mathcal{U}_a^{\mathcal{H}} = \mathbf{Z}_N$) reads

$$S_{((a,i),m)((b,j),m')} = \frac{N}{\sqrt{N^4}}\sum_{n=0}^{N-1}\psi_m^n S_{(a,i)(b,j)}^n(\psi_{m'}^n)^*$$

Comparing this with (4.3) we read off the matrix elements of $S^{J^{\mathcal{H}}}$ (note that the term with $n=1$ in the sum is the generator of the orbit, which corresponds by construction to $J^{\mathcal{H}}$)

$$S_{(a,i)(b,j)}^{J^{\mathcal{H}}} = \frac{|\mathcal{H}|}{\sqrt{|\mathcal{S}_a||\mathcal{U}_a||\mathcal{S}_b||\mathcal{U}_b|}}\Bigg[\sum_{h\in\mathcal{U}_a\cap\mathcal{U}_b}\Psi_i^h S_{ab}^{R_{ab}(J)h}(\Psi_j^h)^*\Bigg] \phi_i(J,a)\Psi_i^{(R_{ab}(J)/R_a(J))}\ [\phi_j(J,b)\Psi_j^{(R_{ab}(J)/R_b(J))}]^*\,. \tag{4.4}$$

This comparison may seem to give us the fixed point resolution matrices for all powers of $J^{\mathcal{H}}$. This, however, is misleading because if N is not prime some powers of $J^{\mathcal{H}}$ may

have additional fixed points not seen here. Obviously the fixed point resolution matrices for powers of $J^{\mathcal{H}}$ is given by the same formula, but with a larger index set (a,i).

Eqn. (4.4) is the main result of this paper. We will now examine some consequences

5. Properties of fixed point resolution matrices

One can go systematically through conditions $\{1\}$ – $\{6\}$ that fixed point resolution matrices must satisfy, but we will omit most of the details here. Condition $\{1\}$ is satisfied by construction, and checking conditions $\{2\}$ and $\{3\}$ is mainly a tedious exercise, similar to the one presented in [1], but with a few extra twists.

Also checking $\{6\}$ is straightforward. One finds furthermore that the matrices S^J – unlike those of unextended WZW models – are in general not symmetric. An example is $A_{4,5}A_{4,5}$ extended with the current (J,J), where J is a simple current of $A_{4,5}$. The new theory has a center $\mathbf{Z}_5$ generated by (the orbit of) $(J,0)$. The fixed point resolution matrices for $(J,0)$, $(J^2,0)$, $(J^3,0)$, $(J^4,0)$ are all different and all of them are asymmetric.

5.1. Simple current twists

Condition $\{4\}$ is of special interest, since it involves a new quantity, the twist $F^{\mathcal{H}}$. The twist $F^{\mathcal{H}}(a^{\mathcal{H}},K^{\mathcal{H}},J^{\mathcal{H}})$ is unambiguously defined only if $K^{\mathcal{H}}$ fixes $a^{\mathcal{H}}$. If we take for $K^{\mathcal{H}}$ a representative $R_a(K)$ that is untwisted w.r.t. $\mathcal{U}_a$ and fixes a we find immediately

$$F^{\mathcal{H}}(a^{\mathcal{H}},K^{\mathcal{H}},J^{\mathcal{H}})=F(a,R_a(K),R_{ab}(J)h)=F(a,R_a(K),R_{ab}(J))$$

note that the result does not depend on h (since $F(a,R_a(K),h)=1$) and hence not on the choice of representative $R_{ab}(J)$.

However, it does appear that $F^{\mathcal{H}}$ now depends on both a and b. Although condition $\{5\}$ looks like merely a definition of F, it does imply the non-trivial requirement that F be independent of b. In addition, for unextended WZW-models F is also independent of a, but this is not required.

Let us therefore carefully examine the a and b dependence of $F^{\mathcal{H}}$. Compare first $F(a,R_a(K),R_{ab}(J))$ and $F(a,R_a(K),R_{ab'}(J))$. Here we made the induction hypothesis that F has no *explicit* dependence on b. The only potential b-dependence is then via the choice of representatives R_{ab}. However, R_{ab} and $R_{ab'}$ differ by an element of $\mathcal{U}_a$, and since $F(a,R_a(K),\mathcal{U}_a)=1$ we have $F(a,R_a(K),R_{ab}(J))=F(a,R_a(K),R_{ab'}(J))$, so that there is no b-dependence. Then the obvious choice is $b=a$, so that (since $R_{aa}(J)=R_a(J)$)

$$F^{\mathcal{H}}(a^{\mathcal{H}},K^{\mathcal{H}},J^{\mathcal{H}})=F(a,R_a(K),R_{aa}(J))=F(a,R_a(K),R_a(J))\ .$$

Now consider the a-dependence. Even if F does not depend on a explicitly via its first argument, there is no reason why an a-dependence could not arise through the choice of

representatives R_a. Indeed, it is easy to find an example. Consider the minimal $N=1$ super conformal field theory realized by the coset CFT $SU(2)_4 \times SU(2)_2/SU(2)_6$. (the supersymmetry plays no rôle here, however). The field identification current is $(4,2,6)$, and there are three non-trivial simple currents, whose orbits are $J=(4,0,0)+(0,6,2)$, $K=(4,0,2)+(0,6,0)$ and $L=(0,0,2)+(4,6,0)$ (the latter is in fact the supercurrent). Consider the field representatives $a=(2,1,1)$ and $a'=(4,3,1)$, which are both fixed by all three currents. The current representatives that actually fix these field representatives are respectively $R_a(J)=(4,0,0)$, $R_a(K)=(4,0,2)$ and $R_a(L)=(0,0,2)$. For a' we find $R_{a'}(J)=(0,6,2)$, $R_{a'}(K)=(0,6,0)$ and $R_{a'}(L)=(0,0,2)$. The twists all originate from the $SU(2)_k$ twists $F(a,J,J)=(-1)^{k/2}$ for k even. It is then easy to see that all twists among J,K and L are opposite for a and a'.

5.2. Conjugation twists

To determine the charge conjugation matrix we need to compute $[S^{J^{\mathcal{H}}}]^2$, from which the coefficients η can be extracted. After a lengthy computation we arrive at the following complicated expression

$$\eta_{a^{\mathcal{H}}}^{J^{\mathcal{H}}} = \eta_{K_a a}^{R_{ac}} F(a,K_a,R_{ac})^* \phi_i(a) (\phi_{\pi^a(i)}(c)^* \ \Psi_i^{R_{ac}/R_a} (\Psi_{\pi^a(i)}^{R_{ac}/R_c})^* \ .$$

The notation is as follows. The conjugate of the orbit $a^{\mathcal{H}}$ is $c^{\mathcal{H}}$, where a and c are arbitrarily chosen representatives. Since they are arbitrarily chosen, they are in general not each other's conjugates, but c^* differs from a by an element of $\mathcal{H}$. This element is denoted as K_a, and its definition is $K_a a = c^*$. Furthermore π^a is some – in general a-dependent – permutation, defined by

$$F(a,K_a,h)^* \eta_{K_0 a}^h \Psi_i^h = \psi_{\pi^a(i)}^h \tag{5.1}$$

This expression can be simplified considerably if we link the choices of representatives on conjugate orbits. If c is conjugate to a, or lies on the same orbit as a^*, a valid choice is certainly

$$R_{ac} = R_a = R_c \ .$$

Then automatically $\phi_i^a = \phi_i^c$. Furthermore it is always possible to choose η in such a way that $\eta_{Ka} = \eta_a$. Now $\eta^{\mathcal{H}}$ simplifies to

$$\eta_a^{J^{\mathcal{H}}} = \eta_a^{R_a(J)} F(a,K_a,R_a(J))^* \phi_i(a,J) \phi_{\pi^a(i)}(a,J)^*$$

This expression may appear to depend on the choice of representatives R_a but actually this is not the case: all four factors depend on R_a, but the overall dependence cancels.

The product formula for two η's is

$$\begin{aligned}\eta_a^{J^{\mathcal{H}}}\eta_a^{L^{\mathcal{H}}} &= \eta_a^{R_a(J)}F(a,K_a,R_a(J))^*\phi_i(a,J)\phi_{\pi^a(i)}(a,J)^* \\ &\eta_a^{R_a(L)}F(a,K_a,R_a(L))^*\phi_i(a,L)\phi_{\pi^a(i)}(a,L)^* \\ &= F(a,R_a(J),R_a(L))\eta_a^{R_a(J)R_a(L)}F(a,K_a,R_a(J)R_a(L)) \\ &\phi_i(a,J)\phi_{\pi^a(i)}(a,J)^*\phi_i(a,L)\phi_{\pi^a(i)}(a,L)^*\end{aligned}$$

Here we used the formula

$$\eta_a^J\eta_a^K = F(a,J,K)\eta_a^{JK}$$

as an induction hypothesis.

Now we may use $R_a(J)R_a(L) = R_a(JL)h(J,L)$. Since the set of representatives was fixed so that $R_a(M)$ fixes a and is not twisted with respect to $\mathcal{S}_a$ (where M can be K,L or KL), and was determined up to elements in $\mathcal{U}_a$, $h(J,L)$ must be an element of $\mathcal{U}_a$. Then

$$\eta_a^{R_a(J)R_a(L)} = F(a,R_a(JL),h(J,L))\eta_a^{R_a(JL)}\eta_a^{h(J,L)} = \eta_a^{R_a(JL)}\eta_a^{h(J,L)}$$

and

$$F(a,K_a,R_a(J)R_a(L)) = F(a,K_a,R_a(JL))F(a,K_a,h(J,L))$$

Furthermore

$$\phi_i(a,J)\phi_i(a,L) = \phi_i(a,JL)\Psi_i^{h(J,K)}$$

The $h(i,j)$-dependent phases all cancel because of (5.1), and we are left with

$$\begin{aligned}\eta_a^{J^{\mathcal{H}}}\eta_a^{L^{\mathcal{H}}} &= F(a,R_a(J),R_a(L))\eta_a^{(JL)^{\mathcal{H}}} \\ &= F^{\mathcal{H}}(a^{\mathcal{H}},J^{\mathcal{H}},L^{\mathcal{H}})\eta_a^{(JL)^{\mathcal{H}}}\end{aligned}$$

This shows that the relation $G = F$ does indeed hold, if it was valid in the unextended theory.

6. Conclusions

We have derived the generalization of the formula of [1] for the modular transformation matrix of the zero-point holomorphic blocks on the torus to modular transformation matrix of the one-point, simple current blocks. In doing so, we have also generalized [1] from unextended WZW-models to extended ones. The main result, formula (4.4), gives the fixed point resolution matrices for all simple current extended WZW-models and – perhaps most importantly – most coset CFT's.

The disclaimer made in [1] regarding these matrices applies here as well: the conditions for S^J are necessary, but not sufficient. In particular fusion rule integrality does not – or at least not manifestly – follow from these conditions. Nevertheless, the fact that these matrices provide natural and general solutions to the non-trivial conditions of [1] may be regarded as strong evidence for their correctness. Formula (4.4) has been checked numerically in many examples, and gives integral fusion rules in all cases that were examined.

In comparison to the unextended WZW-models considered in [1], The matrices given (4.4) satisfy the conditions of [1] in non-trivial ways, exploiting nearly all allowed features, with the interesting exception of complex simple current twists.

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APPENDIX A

Here we show that if $F(a,K,J)$ is real, and the fixed point resolution matrices of tensor products are the tensor products of those of the factors, then $F=G$.

Suppose two closed sets of simple currents in CFT's C and C′ and their twists on fields a and a' are isomorphic in the following sense: their exists a one-to-one map $\pi(J) = J'$ from the currents of C to the currents in C′ which preserves all fusion products, conformal weights modulo integers, and the simple current twists, $i.e. F(a',\pi(J),\pi(K)) = F(a,J,K)$. The currents are assumed to have integral or half-integral spin, and to be local with respect to each other. The group they form under fusion will be called $\mathcal{H}$.

Denote fields in the tensor product $\mathrm{C}\otimes\mathrm{C}'$ as (a,a'). In the tensor product we may consider the set of currents $(J,\pi(J))$. All currents of this form are local with respect to each other, they have integral spin, they form a closed set under fusion and on the field (a,a') their twists cancel. Hence $\mathcal{S}_{(a,a')} = \mathcal{U}_{(a,a')} = \mathcal{H}$. Then in order to satisfy $\{5b\}$ in

the tensor product theory we need

$$G((a,a'),(K,K'),(L,L')) = G(a,K,L)G(a,K',L') = 1$$

Hence knowing $G(a,K,L)$ for one theory, we have now derived $G(a',K',L')$ for any other, isomorphic set of currents. In particular, if for any set of currents and twists we can find an example (a CFT, a set of currents, and a field a) with $F(a,K,L) = G(a,K,L)$ we have proved this relation for any CFT.

Since G and F depend on just two currents, it is clearly sufficient to look only at pairs of currents, *i.e.* at $\mathbf{Z}_N \times \mathbf{Z}_M$. Let us call the generators of the two factors J and K. Denote the spins modulo integers by s_J and s_K. In general we have [1] $F(a,J,J) = (-1)^{2s_J}$. By the properties of twists, all twists are then encoded in the quantity $F(a,J,K) \equiv F$. The following table lists the possibilities and the realization of these possibilities in terms of a tensor product. The factors in the tensor product are denoted A_{N-1} (for A_{N-1} at level N) and $\mathcal{I}$ (for the Ising model). The first two lines describe theories with a single cyclic factor, the remaining ones list all possibilities for $\mathbf{Z}_N \times \mathbf{Z}_M$, assuming F is real. Note that $F(a,K,J) = -1$ requires N (and M) to have even order, since $F(a,K^N,J) = (-1)^N = 1$. For all these theories (2.4) does indeed hold.

s_J	s_K	N	M	F	factors	J	K
0	–	any	–	–	A_{N-1}	J	–
$\frac{1}{2}$	–	even	–	–	$A_{N-1}\,\mathcal{I}$	(J,Ψ)	–
0	0	any	any	1	$A_{N-1}A_{M-1}$	$(J,1)$	$(1,J)$
0	$\frac{1}{2}$	any	even	1	$A_{N-1}A_{M-1}\,\mathcal{I}$	$(J,1,1)$	$(1,J,\Psi)$
$\frac{1}{2}$	$\frac{1}{2}$	even	even	1	$A_{N-1}A_{M-1}\,\mathcal{I}^2$	$(J,1,\Psi,1)$	$(1,J,1,\Psi)$
0	0	even	even	-1	$A_{N-1}A_{M-1}\,\mathcal{I}^3$	$(J,1,\Psi,\Psi,1)$	$(1,J,\Psi,1,\Psi)$
0	$\frac{1}{2}$	even	even	-1	$A_{N-1}A_{M-1}\,\mathcal{I}^2$	$(J,1,\Psi,\Psi)$	$(1,J,\Psi,1)$
$\frac{1}{2}$	$\frac{1}{2}$	even	even	-1	$A_{N-1}A_{M-1}\,\mathcal{I}$	$(J,1,\Psi)$	$(1,J,\Psi)$

Although most of this argument can be extended to complex F, one cannot rule out the possibility that $G(a,J,K) = 1$ for all a, J and K (which is obviously the only way out). However, no examples of this kind are known.

APPENDIX B

Untwisted representatives

Here we will show that if the $\mathrm{C}^{\mathcal{H}}$ current $J^{\mathcal{H}}$ fixes a field $a^{\mathcal{H}}$, one can find an orbit representative $R_a(J)$ in the $\mathcal{H}$-orbit of J such that

$$F(a, R_a(J), K) = 1 \text{ for all } K \in \mathcal{S}_a \ .$$

We drop the explicit dependence on a henceforth. Consider some representative $X(J)$ that fixes a. The choice of $X(J)$ is fixed up to multiplication by elements of $\mathcal{S}$. Consider $\mathcal{S}/\mathcal{U}$. The twist $F(K, L)$ satisfies the group properties

$$\begin{aligned} F(K_1K_2, L) &= F(K_1, L)F(K_2, L) \\ F(K, L_1L_2) &= F(K, L_1)F(K, L_2) \end{aligned}$$

if K, K_1 and K_2 fix a (L, L_1 and L_2 must also fix a, since otherwise F is not defined). Hence

$$F(K\mathcal{U}, L\mathcal{U}) = F(K, L)$$

i.e. the twist is constant on coset classes. Choose a set of generating representatives L_i of $\mathcal{S}/\mathcal{U}$, so that the full set of representatives is of the form $L^{\vec{m}} \equiv \prod_i (L_i)^{m_i}$, $0 < m_i < N_i$, with N_i the smallest positive integer so that $L^{N_i} \in \mathcal{U}$. Then it follows from the product rules of F that the relative twists of the generating representatives has the form

$$F(L_i, L_j) = e^{2\pi i r_{ij}/N_{ij}} \ ,$$

where N_{ij} is the greatest common divisor of N_i and N_j, and r_{ij} a set of integers.

The twist of the $X(J)$ with respect to the generators is

$$F(X(J), L_i) = e^{2\pi i p_i/N_{iJ}} \ .$$

Our claim is now that another representative exists which has trivial twist. Consider $X(J)L^{\vec{k}}$. Then

$$F(X(J)L^{\vec{k}}, L_i) = e^{2\pi i (p_i/N_{iJ} + k_j r_{ji}/N_{ij})}$$

Clearly we need to find a solution $\vec{k}$ to the equation

$$\sum_j k_j \frac{r_{ji}}{N_{ij}} = -\frac{p_i}{N_{iJ}} \mod 1 \tag{B.1}$$

The definition of $\mathcal{S}/\mathcal{U}$ implies that all currents $L^{\vec{m}}$ are twisted with respect at least on other current. This means that $F(L^{\vec{m}}, L_i) = 1$ has no solutions except the trivial one,

$\vec{m}=0$. This implies that

$$m_i \frac{r_{ij}}{N_{ij}} = 0 \bmod 1 \quad \Longleftrightarrow \quad \vec{m}=0 \ . \tag{B.2}$$

Roughly speaking (B.1) has precisely one solution because (B.2) implies that the matrix r_{ij}/N_{ij} is "invertible". But because we work with integers and modulo 1 this is not correct as it stands.

The correct solution to this problem can be found in [14], where exactly the same condition emerges in a quite different situation, although also related to simple currents. In [14] the most general simple current invariant was constructed given a center $\mathbf{Z}_{N_1}\ldots\mathbf{Z}_{N_n}$ with generators $J_1\ldots J_n$. It was found that each invariant was specified by a matrix (called X in the second paper in [14]) of the form r_{ij}/N_{ij}, where again $r_{ij}\in\mathbf{Z}$ and $N_{ij}=\mathrm{GCD}(N_i,N_j)$. In that context, (B.2) is the condition for having a pure automorphism invariant without extension of the chiral algebra. Equation (B.1) appears as the equation determining the way the automorphism acts on a field of charges p_i/N_{iJ}, which are allowed charges w.r.t. the currents J_i. The modular invariant can be written down and checked explicitly, and since it is of automorphism type there is one and only one solution $\vec{k}$ for any $\vec{p}$. This proves not only that the "untwisted" representative exists, but also that it is unique up to multiplication by elements of $\mathcal{U}_a$.

APPENDIX C

Subgroup Characters

Consider a finite, discrete, abelian group $\mathcal{G}$ with subgroup $\mathcal{H}$. The coset $\mathcal{C}=\mathcal{G}/\mathcal{H}$ is also an abelian group. Both $\mathcal{H}$ and $\mathcal{C}$ have a set of characters, which we will denote by Ψ and ψ respectively. We want to express the characters of $\mathcal{G}$ in terms of Ψ and ψ.

We denote the elements of $\mathcal{H}$ as h, and the character labels as $i,j,\ldots$. The characters will be denoted as Ψ_i^h. Being characters, they satisfy the group properties

$$\Psi_i^h\Psi_i^g=\Psi_i^{hg}\ , \quad \Psi_i^1=1\ .$$

Furthermore they satisfy an orthogonality relation

$$\sum_h \Psi_i^h(\Psi_j^h)^* = |\mathcal{H}|\delta^{ij}$$

and a completeness relation

$$\sum_i \Psi_i^h(\Psi_i^g)^* = |\mathcal{H}|\delta^{hg}\ .$$

The labels i are not assumed to be ordered in any particular way.

The elements of $\mathcal{C}$ are denoted as J, and the character labels are denoted $m, n, \ldots$. The characters are thus denoted ψ_m^J, and have properties analogous to those of Ψ. In each $\mathcal{G}/\mathcal{H}$ coset J we choose a representative $R(J) \in \mathcal{G}$. The elements of $\mathcal{G}$ are all of the form $R(J)h$, for all $J \in \mathcal{C}$ and all $h \in \mathcal{H}$. In general the set of representatives only closes up to elements of $\mathcal{H}$:

$$R(J)R(K) = R(JK)h(J,K) \ , \tag{C.1}$$

with $h(J,K) \in \mathcal{H}$.

A complete set of character labels is (m,i) where m and i are from the same index sets as before. This ensures that we get at least the right number of labels. An obvious guess for the characters Ξ of $\mathcal{G}$ could be to take simply the product of those of $\mathcal{H}$ and $\mathcal{G}/\mathcal{H}$, but it is easy to see that in general this yields incorrect group properties, for any choice of representatives. To remedy this we try the following ansatz

$$\Xi_{(m,i)}^{R(J)h} = \psi_m^J \Psi_i^h \phi_i^J \ ,$$

where ϕ is assumed to be a phase (it is actually a cocycle – see the appendix of the third paper of [11] for a related discussion).

Let us first demonstrate orthogonality and completeness of these characters.

— Orthogonality:

$$\sum_{m,i} \Xi_{(m,i)}^{R(J)h} [\Xi_{(m,i)}^{R(J')h'}]^* = \sum_{m,i} \psi_m^J \Psi_i^h \phi_i^J [\psi_m^{J'} \Psi_i^{h'} \phi_i^{J'}]^*$$

The sum over m can be done and yields (up to normalization) $\delta_{JJ'}$ using orthogonality in $\mathcal{C}$. Then the phases ϕ cancel, and we can perform the sum over i.

— Completeness:

$$\sum_{J,h} \Xi_{(m,i)}^{R(J)h} [\Xi_{(m',i')}^{R(J)h}]^* = \sum_{J,h} \psi_m^J \Psi_i^h \phi_i^J [\psi_{m'}^J \Psi_{i'}^h \phi_i^J]^*$$

The idea is similar. Now we can sum in $\mathcal{H}$ over h to get $\delta_{ii'}$, the phases ϕ cancel, and we then sum over J.

Note that it is essential in these arguments that ϕ does not depend on on h and m. Clearly orthogonality and completeness do not fix ϕ. Consider now the group property.

On the one hand

$$\Xi^{R(J)h}_{(m,i)}\Xi^{R(K)g}_{(m,i)} = \Xi^{R(JK)h(J,K)hg}_{(m,i)} = \psi^{JK}_m \Psi^{h(J,K)hg}_i \phi^{JK}_i$$

On the other hand the product is

$$\psi^J_m \Psi^h_i \phi^J_i \psi^K_m \Psi^g_i \phi^K_i = \psi^{JK}_m \Psi^{hg}_i \phi^J_i \phi^K_i$$

Comparing the two expressions we find

$$\Psi^{h(J,K)}_i \phi^{JK}_i = \phi^J_i \phi^K_i \tag{C.2}$$

It is easy to show that a change in representatives in the cosets can be compensated by a change in ϕ. This observation helps in determining the phases ϕ, because it allows us to make a convenient choice of representatives. For the identity we choose representative 1. Choose a basis in $\mathcal{C}$ (by “basis” we mean here a set of elements $J_1, \ldots, J_n$ generating cyclic subgroups, so that any other element can be written as $(J_1)^{m_1} \ldots (J_n)^{m_n}$). Consider one such cyclic subgroup of order N, and denote the generator J_ℓ simply as J. If $R(J)$ is a representative of class J, then $R(J)^m$ is a representative of class J^m. Within the cyclic subgroup we have then

$$R(J^m)R(J^n) = R(J^{m+n})$$

as long as $m+n < N$, the order of J. In general $R(J)^N = h_J \in \mathcal{H}$. Then we find

$$h(m,n) \equiv h(J^m, J^n) = 1 \ \text{ if } m+n<N; \quad h(J^m,J^n) = h_J \ \text{ otherwise}$$

(note that $m+n < 2N$). Now we can fix the representatives on all other elements as

$$R((J_1)^{m_1} \ldots (J_n)^{m_n}) = R(J_1)^{m_1} \ldots R(J_n)^{m_n}$$

The computation of the function h is then as follows

$$\begin{aligned} &R((J_1)^{m_1} \ldots (J_n)^{m_n}) R((J_1)^{k_1} \ldots (J_n)^{k_n}) \\ &\qquad = R((J_1)^{m_1+k_1} \ldots (J_n)^{m_n+k_n}) h_1(m_1,k_1) \ldots h_n(m_n,k_n) \end{aligned}$$

Now we determine the phases ϕ. On one cyclic subgroup of order N, and generated

by J, we have

$$\begin{aligned}
\phi_i^{J^2} &= \phi_i^J \phi_i^J \\
\phi_i^{J^3} &= \phi_i^{J^2} \phi_i^J = (\phi_i^J)^3 \\
&\dots \\
\phi_i^{J^{N-1}} &= \phi_i^{J^{N-2}} \phi_i^J = (\phi_i^J)^{N-1} \\
\phi_i^{J^N} &= \phi_i^{J^{N-1}} \phi_i^J [\Psi_i^{h_J}]^* = (\phi_i^J)^N [\Psi_i^{h_J}]^*
\end{aligned}$$

On the other hand, since $J^N = 1$ and the phases ϕ depend only on the elements of $\mathcal{C}$, we must demand that

$$\phi_i^{J^N} = 1$$

This the implies

$$\phi_i^J = [\Psi_i^{h_J}]^{1/N} \ ,$$

i.e. it should be equal to one of the N^{th} roots of $\Psi_i^{h_J}$. It should not matter which root we choose, since different choices amount only to permutations of labels m on the $\mathcal{G}$ characters. Indeed, the different roots are

$$[\Psi_i^{h_J}]^{1/N} \psi_m^J \quad m = 0 \dots, N-1 \ .$$

With this definition the phases on each cyclic subgroup satisfy

$$\phi_i^{J^m} \phi_i^{J^n} = \phi_i^{J^{h+n}} \Psi_i^{h(m,n)} \ ,$$

with $h(m,n)$ as defined above. This procedure is repeated for each cyclic subgroup generated by the basis elements J_ℓ. To keep the notation manageable we define the phases in the ℓ^{th} cyclic factor as

$$\phi_i^m(\ell) \equiv \phi_i^{J_\ell^m}$$

Having in this manner determined the phases on the cyclic subgroups of the individual generators, we have for the general case

$$\phi_i^{m_1,\dots,m_n} = \prod_{\ell=1}^{n} \phi_i^{m_\ell}(\ell)$$

The final check is then the product rule

$$\phi_i^{\vec{m}} \phi_i^{\vec{n}} = \prod_{\ell=1}^{n} \phi_i^{m_\ell}(\ell) \phi_i^{n_\ell}(\ell) = \prod_{\ell=1}^{n} \phi_i^{m_\ell+n_\ell} \Psi_i^{h_\ell(m_\ell,n_\ell)} = \phi_i^{\vec{m}+\vec{n}} \Psi_i^{h(\vec{m},\vec{n})}$$

with the understanding that $\vec{m} + \vec{n}$ is taken modulo $\vec{N}$.

Having determined ϕ for a convenient set of representatives R, we can transform it to any other set in the following way. Suppose we are given a any other set of representatives (with the canonical choice on the identity) $r(\vec{m})$. Then $r(\vec{m}) = H(\vec{m})R(\vec{m})$, with $H(\vec{m}) \in \mathcal{H}$, and R is the choice used above. Then the phases ϕ with respect to the r-representatives are

$$\phi_i^{\vec{m}}(r) = \phi_i^{\vec{m}}(R)\Psi_i^{H(\vec{m})} \ .$$

It is straightforward to check that these phases ϕ do indeed satisfy (C.2), with $h(J,K)$ determined from $r(\vec{m})$ using (C.1).

Note that this character decomposition has the property that it reduces to $\mathcal{H}$ characters for elements of $\mathcal{H}$, provided we choose $R(1) = 1$. Then $R(1)R(1) = R(1)$, hence $h(1,1) = 1$ and hence $\phi_i^1\phi_i^1 = \phi_i^1$ so that $\phi_i^1 = 1$.

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